projective representations of finite groups

projective representations of finite groups are an important concept in the intersection of group theory and linear algebra, particularly relevant in mathematical physics, representation theory, and algebraic topology. These representations generalize the notion of linear representations by allowing group elements to be represented by linear operators up to a scalar factor, capturing more nuanced symmetries than ordinary group representations. This article provides a thorough introduction to projective representations of finite groups, exploring their definitions, construction, properties, and applications. It also discusses the role of cocycles and Schur multipliers in classifying these representations, alongside examples illustrating their significance. The connections between projective representations and central extensions of groups further enrich the theoretical framework. The following table of contents outlines the key topics covered.

Definition and Basic Concepts

Projective representations of finite groups extend the classical concept of group representations by relaxing the requirement that the group homomorphism must strictly preserve the group multiplication law. Formally, a projective representation of a finite group G is a map from G to the projective linear group PGL(V), where V is a vector space over a field (commonly the complex numbers). Unlike ordinary representations, which map group elements to invertible linear operators satisfying the homomorphism property exactly, projective representations satisfy this property up to a scalar factor. More precisely, for a projective representation □ of G, the product of images satisfies □(g)□(h) = c(g,h)□(gh) for all g,h in G, where c(g,h) is a nonzero scalar called a factor set or cocycle. This scalar ambiguity encodes additional algebraic structure that cannot be captured by linear representations alone. Projective representations are often studied through their associated factor sets and equivalence classes, providing a richer framework for understanding group symmetries.

Linear vs. Projective Representations

While linear representations map group elements directly to linear transformations respecting group multiplication, projective representations allow the multiplication to hold up to a scalar factor. This subtle difference enables projective representations to classify symmetries that linear representations cannot, particularly in quantum mechanics and other areas of physics where phase factors are physically indistinguishable.

Mathematical Setting

Typically, projective representations are considered over the complex numbers, allowing the use of unitary operators and Hilbert spaces. The scalar factors form a 2-cocycle taking values in the multiplicative group of nonzero complex numbers, which lies at the heart of the cohomological classification of projective representations.

Cocycles and Factor Sets

Central to the theory of projective representations of finite groups are cocycles and factor sets, which characterize the obstruction to lifting a projective representation to an ordinary linear representation. A 2-cocycle is a function $c: G \times G \to C^\times$ satisfying the cocycle condition:

$c(g,h)c(gh,k) = c(h,k)c(g,hk)$ for all g,h,k in G.

This condition ensures associativity in the projective representation up to scalar factors. The function c is also called a factor set or multiplier and encodes how the projective representation deviates from a genuine group homomorphism.

Properties of Factor Sets

Factor sets must satisfy normalization conditions, often chosen so that $c(e,g) = c(g,e) = 1$, where e is the identity element of G. Two factor sets are considered equivalent if they differ by a coboundary, a function arising from a change of basis in the vector space V. This equivalence is fundamental in classifying projective representations up to isomorphism.

Cohomology Interpretation

The set of equivalence classes of factor sets forms the second cohomology group $H^2(G, C^\times)$, a central object in the classification of projective representations. The trivial cohomology class corresponds to linear representations, while nontrivial classes correspond to genuinely projective representations that cannot be lifted to linear ones.

Schur Multipliers and Classification

The Schur multiplier is a crucial invariant in the study of projective representations of finite groups. It arises as the second cohomology group $H^2(G, C^\times)$ and provides a measure of the "projective complexity" of a group. The Schur multiplier classifies all possible projective representations up to equivalence and plays a key role in understanding the structure of central extensions.

Definition and Role

Named after Issai Schur, the Schur multiplier is an abelian group associated with a finite group G. It quantifies the obstructions to lifting projective representations to linear representations by capturing all possible factor sets modulo coboundaries. The size and structure of the Schur multiplier give insight into the diversity of projective representations available for a given group.

Computing Schur Multipliers

Computing the Schur multiplier generally involves group cohomology techniques and can be complex for arbitrary groups. However, for many finite groups of interest, such as symmetric, alternating, or cyclic groups, the Schur multiplier is well-understood and tabulated. Knowledge of the Schur multiplier is essential for constructing and classifying projective representations.

Relation to Central Extensions

Projective representations of finite groups are intimately connected to central extensions of groups. A central extension of a group G by an abelian group A is a group E fitting into a short exact sequence:

$$1 \to A \to E \to G \to 1$$

These extensions provide a mechanism to "lift" projective representations of G to linear representations of E, thereby resolving the scalar ambiguity inherent in projective representations.

From Projective Representations to Central Extensions

Every projective representation of G with factor set c corresponds to a central extension E of G by $C^\times$, where the multiplication in E encodes the cocycle c. The projective representation then lifts to an ordinary representation of E, allowing the use of classical representation theory tools. This lifting is a powerful method for studying projective representations via linear representations of larger groups.

Classification via Extensions

The second cohomology group $H^2(G, C^\times)$, which classifies factor sets, also classifies equivalence classes of central extensions of G by $C^\times$. This duality provides a deep link between algebraic topology, cohomology, and group theory, facilitating the understanding and construction of projective representations through central extensions.

Examples of Projective Representations

Examining concrete examples helps illustrate the abstract theory of projective representations of finite groups. Well-known groups such as the cyclic group, symmetric groups, and dihedral groups exhibit interesting projective representations with nontrivial factor sets and Schur multipliers.

Cyclic Groups

For cyclic groups, projective representations are relatively straightforward since their second cohomology groups are often trivial or cyclic. This simplicity allows explicit computation of factor sets and easy classification of projective representations.

Symmetric and Alternating Groups

Projective representations of symmetric and alternating groups are of particular interest, especially in mathematical physics. The Schur multiplier of the symmetric group S_n is nontrivial for $n \geq 4$, leading to nontrivial projective representations. These representations play a role in the theory of spin representations and the study of fermionic symmetries.

Dihedral Groups

Dihedral groups serve as classical examples where projective representations can be constructed explicitly. The structure of their Schur multipliers allows for a variety of projective representations, illustrating how factor sets arise naturally in group extensions.

Applications in Mathematics and Physics

Projective representations of finite groups have broad applications across various fields of mathematics and theoretical physics. They provide a framework for understanding symmetries that are not captured by linear representations alone, especially in contexts where phase factors or central charges are significant.

Quantum Mechanics and Particle Physics

In quantum mechanics, physical states are defined up to a global phase, making projective representations the natural language for describing symmetry groups of quantum systems. Spinor representations and the classification of elementary particles often rely on projective representations and their associated central extensions, such as the spin groups.

Algebraic Topology and Cohomology

The cohomological classification of projective representations connects deeply with algebraic topology. The study of group cohomology, characteristic classes, and fiber bundles often employs projective representations to capture topological and geometric features of group actions.

Mathematical Representation Theory

Within pure mathematics, projective representations enrich the representation theory of finite groups by providing a broader category of representations. This leads to a more complete understanding of group symmetries, module theory, and the structure of group algebras.

Summary of Key Applications

- Modeling quantum state symmetries and phase factors
- Construction of spin and fermionic representations
- Classification of central extensions and Schur covers
- Connections with group cohomology and algebraic topology
- Enhancement of classical representation theory frameworks

Frequently Asked Questions

What is a projective representation of a finite group?

A projective representation of a finite group G is a map from G to the projective linear group PGL(V), or equivalently, a map from G to GL(V) that satisfies the group multiplication up to a scalar factor. Formally, it is a function $\rho: G \to GL(V)$ such that $\rho(g)\rho(h) = c(g,h) \rho(gh)$ for some scalar $c(g,h)$ in the field, usually the complex numbers.

How are projective representations related to group cohomology?

Projective representations of a finite group G correspond to 2-cocycles in the second cohomology group $H^2(G, \mathbb{C}^*)$. The scalar factors $c(g,h)$ form a 2-cocycle, and different projective representations correspond to different cohomology classes. This connection allows classification of projective representations via group cohomology.

What is Schur's multiplier and its role in projective representations?

Schur's multiplier is the second cohomology group $H^2(G, \mathbb{C}^*)$ of a finite group G. It measures the obstruction to lifting projective representations to genuine linear representations of a central extension of G. The Schur multiplier classifies the possible projective representations of G.

Can every projective representation be lifted to a linear representation?

Not every projective representation of a finite group can be lifted to an ordinary linear representation of the group itself. However, every projective representation can be lifted to a linear representation of a suitable central extension of the group, known as the Schur cover.

What are some applications of projective representations of finite groups?

Projective representations appear in various areas including quantum mechanics (where symmetry operations may be represented projectively), the theory of modular forms, and the study of central extensions of groups. They also play a role in the classification of finite simple groups and in condensed matter physics, particularly in the study of symmetry-protected topological phases.

Additional Resources

1. *Projective Representations of Finite Groups* by G. Karpilovsky

This book offers an in-depth treatment of projective representations of finite groups, exploring the connections with cohomology and Schur multipliers. It covers the foundational aspects as well as advanced topics, making it suitable for both beginners and researchers. The text includes numerous examples and exercises to facilitate understanding.

2. *Linear Representations of Finite Groups* by Jean-Pierre Serre

While primarily focused on linear representations, this classic text contains essential insights into projective representations and their role in group theory. Serre introduces cohomological methods that underpin the study of projective representations. The book is concise, clear, and highly influential in the field.

3. *Projective Representations of Symmetric Groups: Q-Functions and Shifted Tableaux* by A. M. Vershik and A. N. Sergeev

This book examines projective representations in the context of symmetric groups, highlighting combinatorial approaches through shifted tableaux and Q-functions. It bridges algebraic and combinatorial methods, providing a specialized but accessible resource. The work has applications in algebraic combinatorics and representation theory.

4. *Cohomology of Groups* by Kenneth S. Brown

Brown's text is fundamental for understanding the cohomological aspects underlying projective representations. It systematically develops group cohomology theory, which is essential for classifying projective representations via Schur multipliers. The book is rigorous and includes many examples relevant to finite groups.

5. *Representation Theory: A First Course* by William Fulton and Joe Harris

This introductory book covers a broad range of representation theory topics, including projective representations in the context of finite groups. It provides clear explanations, geometric intuition, and numerous illustrations. The text is well-suited for graduate students beginning their study of representation theory.

6. *Projective Representations and Covering Groups of Finite Groups* by I. M. Isaacs

Isaacs delves into the theory of projective representations and their relationship with central extensions and covering groups. The book explores the construction and classification of projective representations using group cohomology. It is a valuable resource for understanding the structural aspects of finite groups.

7. *Modular Representations of Finite Groups* by J. L. Alperin

Though focused on modular representations, this book touches on projective modules and projective representations in modular settings. Alperin’s presentation includes discussions on how projective representations behave over fields of positive characteristic. It is a key reference for those interested in modular representation theory.

8. *Schur Algebras and Representation Theory* by Stephen Donkin

Donkin’s work addresses the connections between Schur algebras and projective representations, especially in relation to symmetric groups and general linear groups. The text explores how Schur algebras facilitate the study of projective representations and their applications. It is suited for advanced readers with background in algebra.

9. *Introduction to the Representation Theory of Compact and Locally Compact Groups* by Alain Robert

This book provides a broader context for projective representations by covering representations of more general groups, including compact and locally compact groups. It includes discussions on projective unitary representations and their classification. The comprehensive approach makes it useful for readers interested in both finite and infinite group representations.

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