

principle of mathematical induction

example problems

Principle of Mathematical Induction is a fundamental method of mathematical proof used to establish the validity of a statement for all natural numbers. This principle allows mathematicians to prove propositions that are indexed by the natural numbers, making it an essential tool in various fields such as number theory, combinatorics, and algebra. In this article, we will explore the principle of mathematical induction, its steps, and provide example problems to illustrate how it works.

Understanding the Principle of Mathematical Induction

Mathematical induction consists of two primary steps that demonstrate the truth of an infinite sequence of statements. These steps include:

1. Base Case

The base case involves proving that the statement holds for the initial value, usually $(n = 1)$. This serves as the foundation for the induction process.

2. Inductive Step

In the inductive step, we assume the statement is true for some arbitrary natural number (k) (this assumption is called the inductive hypothesis). We then need to prove that if the statement holds for $(n = k)$, it also holds for $(n = k + 1)$.

If both steps are successfully completed, we can conclude that the statement is true for all natural numbers starting from the base case.

Example Problems Using Mathematical Induction

Let's delve into some example problems to better understand how the principle of mathematical induction can be applied.

Example Problem 1: Sum of the First (n) Natural

Numbers

Statement: Prove that the sum of the first (n) natural numbers is given by the formula:

$$S(n) = 1 + 2 + 3 + \dots + n = \frac{n(n + 1)}{2}$$

Step 1: Base Case

For $(n = 1)$:

$$S(1) = 1 = \frac{1(1 + 1)}{2} = 1$$

The base case holds true.

Step 2: Inductive Step

Assume the statement is true for $(n = k)$:

$$S(k) = \frac{k(k + 1)}{2}$$

We need to prove it for $(n = k + 1)$:

$$S(k + 1) = S(k) + (k + 1) = \frac{k(k + 1)}{2} + (k + 1)$$

To combine the terms:

$$S(k + 1) = \frac{k(k + 1)}{2} + \frac{2(k + 1)}{2} = \frac{k(k + 1) + 2(k + 1)}{2}$$

Factoring out $(k + 1)$:

$$S(k + 1) = \frac{(k + 1)(k + 2)}{2}$$

Thus, the statement holds for $(n = k + 1)$.

Since both steps are confirmed, by the principle of mathematical induction, the statement is true for all natural numbers (n) .

Example Problem 2: Inequality Involving Natural Numbers

Statement: Prove that for all $(n \geq 1)$:

$$\begin{aligned} &[\\ &2^n > n^2 \\ &] \end{aligned}$$

Step 1: Base Case

For $(n = 1)$:

$$\begin{aligned} &[\\ &2^1 = 2 > 1^2 = 1 \\ &] \end{aligned}$$

The base case holds true.

Step 2: Inductive Step

Assume the statement is true for $(n = k)$:

$$\begin{aligned} &[\\ &2^k > k^2 \\ &] \end{aligned}$$

We want to prove it for $(n = k + 1)$:

$$\begin{aligned} &[\\ &2^{k+1} = 2 \cdot 2^k \\ &] \end{aligned}$$

Using the inductive hypothesis:

$$\begin{aligned} &[\\ &2^{k+1} > 2 \cdot k^2 \\ &] \end{aligned}$$

Now we need to show:

$$\begin{aligned} &[\\ &2 \cdot k^2 > (k + 1)^2 \\ &] \end{aligned}$$

Expanding the right side:

$$\begin{aligned} &[\\ &2k^2 > k^2 + 2k + 1 \\ &] \end{aligned}$$

This simplifies to:

$$k^2 - 2k - 1 > 0$$

To find the roots of $(k^2 - 2k - 1 = 0)$:

$$k = \frac{2 \pm \sqrt{4 + 4}}{2} = 1 \pm \sqrt{2}$$

The positive root is approximately (2.41) . Thus, for $(k \geq 3)$, $(k^2 - 2k - 1 > 0)$. Therefore, the inequality holds for $(n = k + 1)$.

This proves the statement holds for all $(n \geq 1)$.

Example Problem 3: Factorial Inequality

Statement: Prove that for all $(n \geq 5)$:

$$n! > 2^n$$

Step 1: Base Case

For $(n = 5)$:

$$5! = 120 > 32 = 2^5$$

The base case holds true.

Step 2: Inductive Step

Assume $(n! > 2^n)$ is true for $(n = k)$:

$$k! > 2^k$$

We need to prove it for $(n = k + 1)$:

$$(k + 1)! = (k + 1) \cdot k!$$

Using the inductive hypothesis:

$$(k + 1)! > (k + 1) \cdot 2^k$$

We need to show:

$$(k + 1) \cdot 2^k > 2^{k+1}$$

This simplifies to:

$$k + 1 > 2$$

This is true for $(k \geq 1)$. Therefore, for $(k \geq 5)$, the inequality holds, proving the statement for all $(n \geq 5)$.

Conclusion

The principle of mathematical induction is a powerful tool for proving statements about natural numbers. Through the systematic process of establishing a base case and proving the inductive step, we can validate a wide range of mathematical propositions. The examples provided demonstrate how induction can be applied effectively to establish the truth of statements, from simple sums to complex inequalities. As you practice and encounter various problems, you will find that mathematical induction is not only a method of proof but also a fascinating approach to understanding the foundations of mathematics.

Frequently Asked Questions

What is the principle of mathematical induction?

The principle of mathematical induction is a proof technique used to establish the truth of an infinite sequence of statements. It consists of two main steps: the base case, where you prove the statement for the initial value (usually $n=1$), and the inductive step, where you assume the statement is true for $n=k$ and then prove it for $n=k+1$.

Can you provide an example problem using mathematical induction?

Sure! Prove that the sum of the first n natural numbers, $S(n) = 1 + 2 + \dots + n$, is equal to $n(n+1)/2$. For the base case $n=1$, $S(1) = 1$, which equals $1(1+1)/2$. For the inductive step, assume $S(k) = k(k+1)/2$ is true. Then for $n=k+1$, $S(k+1) = S(k) + (k+1) = k(k+1)/2 + (k+1)$

$= (k+1)(k/2 + 1) = (k+1)(k+2)/2$, proving it holds for $n=k+1$.

What are some common mistakes when using mathematical induction?

Common mistakes include failing to verify the base case, incorrectly applying the inductive hypothesis, or making algebraic errors when transitioning from $n=k$ to $n=k+1$. It's essential to be meticulous in each step of the proof.

How do you know when to use mathematical induction?

Mathematical induction is particularly useful for proving statements about integers, especially those that involve sequences, sums, or inequalities. If a problem involves a property that must hold for all natural numbers or a specific range of integers, induction is often a suitable approach.

Are there any limitations to using mathematical induction?

Yes, mathematical induction can only be applied to statements involving natural numbers or integers. Additionally, it requires that the statement can be expressed in a way that lends itself to induction. Some problems may be better suited for direct proofs or other proof techniques.

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